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TIME-OPTIMAL STEERING OF A POINT MASS ONTO THE SURFACE OF A SPHERE AT ZERO VELOCITY†

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The problem of the time-optimal steering of a point mass onto the surface of a sphere at zero velocity, by a control force of bounded magnitude is investigated. It is assumed that the surface is penetrable and that the point may "land" on the sphere either from the outside or from the inside. An optimal control, in the open-loop and feedback form of trajectories the optimal time and the Bellman function are constructed using Pontrya'gin's maximum principle. The multidimensional boundary-value problem is reduced, by introducing self-similar variables, to the numerical solution of an algebraic equation of degree four and a transcendental equation. It is shown that the boundary-value problem degenerates when the optimal trajectory is nearly linear; a solution of the synthesis problem is constructed in the degenerate case. The efficacy of the approach proposed here is illustrated by specific examples in which families of trajectories are computed, and by an analysis of control regimes. © 2002 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Consider the motion of a point mass of constant mass m in a space of arbitrary dimensions R^n, $n \geq 1$, driven by a force F of bounded magnitude [1–3]:

$$\dot{x} = v, \quad m\dot{v} = F, \quad x(0) = x^0, \quad v(0) = v^0, \quad |F| \leq F_0 \tag{1.1}$$

The problem is to steer the point in a time-optimal manner to a sphere S_r^n, of arbitrary radius $r \geq 0$ at zero velocity (a "soft landing"):

$$|x(t_f) - x_0| = r, \quad v(t_f) = 0, \quad t_f \to \min_F \tag{1.2}$$

No restrictions are imposed on the possible positions of the point x: it may be located either inside or outside the sphere, and the trajectory may intersect the sphere S_r^n. The situation in which the sphere is "impenetrable" is also of considerable interest, and requires separate consideration. We mention that for a certain set of initial data the solution of problem (1.1), (1.2) also yields a solution of the problem with the indicated state constraints (a "soft landing" from outside or inside). The case $r = 0$ is degenerate and corresponds to steering the point to a geometric point $x(t_f) = x_0$ at zero velocity; a complete solution of this time-optimal problem has already been constructed [3]. We will therefore assume from now on that $r > 0$.

We note that the problem of the most rapid steering of a point to a cylindrical surface $S_r^m \times R^{n-m}$, $m < n$, reduces to problem (1.1), (1.2) when $n = m$.

The formulation of the control problem (1.1), (1.2) and its complete solution in the open-loop and feedback form are of definite interest for applications to the mechanics of flight.

The time-optimal control problem contains $3n + 3$ parameters: the n-vectors x^0, x_0, v^0 and the scalars m, F_0, r. By transferring the coordinate system x, v to the point $x = x_0$, $v = 0$, introducing r as the unit of length and $(mr/F_0)^{1/2}$ as the unit of time, we obtain equations of type (1.1), (1.2) in which $x_0 = 0$, $r = 1$, $F_0 = 1$, $m = 1$. In dimensionless variables, the control problem contains $2n$ parameters: the n-vectors x^0 and v^0, which vary in infinite intervals.

We will write the necessary conditions for the optimality of the control $u = F/F_0$ as a Cauchy problem for the Hamilton–Jacobi–Bellman equation [1]

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$$(T'_x, v) - |T'_v| = -1, \quad u^* = -T'_v |T'_v|^{-1}$$
$$T = T(x, v) > 0, \quad x \overline{\in} S_1^n, \quad v \neq 0; \quad T = 0, \quad x \in S_1^n, \quad v = 0 \tag{1.3}$$

where T is the Bellman function and u^* is an optimal control in the feedback form. The unknown $T(x, v)$ is constructed by the method of characteristics, which is algorithmically equivalent to solving the boundary-value problem of Pontryagin's maximum principle [1]. It follows from the central symmetry property that if $n \geqslant 2$ (the case $n = 1$ is singular and needs special consideration), the unknown function T is defined by three self-similar variables l, h and c, and Cauchy problem (1.3) becomes

$$T'_l c/l + T'_c h^2 - (T'^2_h + 2T'_h T'_c c/h + T'^2_c l^2)^{1/2} = -1, \quad T = T(l, h, c)$$
$$l = |x|, \quad h = |v|, \quad c = (x, v); \quad T > 0, \quad l \neq 1, \quad h \neq 0; \quad T = 0, \quad l = 1, \quad h = 0 \tag{1.4}$$

This means that the time-optimal problem for $n \geqslant 2$ is equivalent to the case $n = 2$, that is, to the two-dimensional problem. The plane in which the optimally controlled motion takes place is defined by the two non-zero vectors x and v. Obviously, the case $c = \pm l h$, including also the case $x = 0$ and/or $v = 0$, leads to degeneration of the one-dimensional motion. The property (1.4) of equivalence to the two-dimensional problem is useful for constructing a solution of the initial multidimensional problem ($n > 2$) of the time-optimal "soft landing" on a sphere. This property also manifests itself when the necessary optimality conditions are applied in the form of Pontryagin's maximum principle [3].

Note that the control problem has a solution for arbitrary values of the vectors x^0 and v^0. As admissible control one can take a constant control $u_{(1)} = -v^0/h^0$, $h^0 = |v^0|$ in the first interval, which leads to a complete halt at time $t_{(1)} = h^0$ at the point $x(t_{(1)}) = x^{(1)}$, where $x^{(1)} = x^0 + \frac{1}{2} v^0 h^0$. One can then move from the rest point along a straight line through a selected point of the sphere, say the closest point, $x^{(2)} = x^{(1)}/l^{(1)}$, $l^{(1)} = |x^{(1)}|$; the distance between the points $x^{(1)}$ and $x^{(2)}$ is $\Delta l_2 = |l^{(1)} - 1|$. Movement along that line requires a time $t_2 = 2\Delta l_2^{1/2}$ to reach the sphere at the point $x^{(2)}$ with zero velocity, that is, we have an upper bound t^* for t_f.

$$t_f \leqslant t^* = h^0 + 2\left|\left(l^{02} + c^0 h^0 + \frac{1}{4} h^{04}\right)^{1/2} - 1\right|^{1/2}$$

Computations (see Section 6) indicate that, for initial data that lead to trajectories with re-entry, the above control mode is approximately realized.

2. THE BOUNDARY-VALUE PROBLEM OF THE MAXIMUM PRINCIPLE

Let us apply the necessary conditions for optimality of the control u, in the form of the maximum principle [3], to the initial problem. Introducing the n-vectors p and q of variables conjugate to x and v, respectively, and proceeding in the usual way, we obtain expressions for the optimal control u^* and the following boundary-value problem [3]

$$u^* = q|q|^{-1}, \quad q = \eta - \mu\nu t/t_f, \quad |q| = (1 - 2\sigma\mu t/t_f + \mu^2 (t/t_f)^2)^{1/2}$$
$$\dot{x} = v, \quad \dot{v} = u^*, \quad x(0) = x^0, \quad v(0) = v^0, \quad |x(t_f)| = 1, \quad v(t_f) = 0 \tag{2.1}$$
$$\mu \geqslant 0, \quad |\sigma| \leqslant 1, \quad \nu = p|p|^{-1}, \quad p = \text{const}, \quad \sigma = (\eta, \nu)$$

where η and ν are unit n-vectors and σ is their scalar product. Note that the Hamiltonian is constant along the trajectories of system (2.1) and is equal to $|\eta - \mu\nu| \geqslant 0$. The unknown scalar parameters μ, σ and t_f and the vectors η and ν must be determined from the boundary (final) conditions and the transversality conditions [1].

To that end, we integrate the equations for ν and x according to (2.1)

$$
\begin{aligned}
v(t) &= v^0 + V_\eta(t)\eta + \mu V_\nu(t)\nu \\
x(t) &= x^0 + v^0 t + X_\eta(t)\eta + \mu X_\nu(t)\nu
\end{aligned}
\tag{2.2}
$$

The scalar functions $V_{\eta,\nu}(t)$ and $X_{\eta,\nu}(t)$ also depend on the unknown parameters μ, σ and t_f. Taking the constraints on t_f, μ and σ into consideration, we express these functions as follows [3]:

$$
\begin{aligned}
&V_\eta(t) = \frac{t_f}{\mu} V(\tau)\Big|_{\tau=0}^{\tau=t}, \quad V_\nu(t) = -\frac{t_f}{\mu^2}[|q(\tau)| + \sigma V(\tau)]_{\tau=0}^{\tau=t} \\
&X_\eta(t) = V_\eta(t)t - \frac{t_f^2}{\mu^2}[|q(\tau)| + \sigma V(\tau)]_{\tau=0}^{\tau=t}, \quad V(t) = \operatorname{arsh}\frac{\mu t/t_f - \sigma}{\sqrt{1-\sigma^2}} \\
&X_\nu(t) = V_\nu(t)t + \left[\frac{t_f^2}{2\mu^3}(\mu\tau/t_f + 3\sigma)|q(\tau)| + \frac{t_f^2}{2\mu^3}(3\sigma^2 - 1)V(\tau)\right]_{\tau=0}^{\tau=t}
\end{aligned}
\tag{2.3}
$$

Substituting formulae (2.3) into (2.2), we obtain the required functions of t and of the vectors η and ν, with t_f, μ and σ occurring as unknowns. Using the appropriate final conditions for $v(t)$ and the transversality conditions of the maximum principle [1] for $x(t)$ at $t = t_f$, we obtain a system of $2n$ equations containing the unknown vector $x(t_f)$

$$
\begin{aligned}
&V_\eta(t_f)\eta + \mu V_\nu(t_f)\nu = -v^0 \\
&X_\eta(t_f)\eta + \mu X_\nu(t_f)\nu = -x^0 - v^0 t_f + x(t_f) \\
&\beta x(t_f) = \nu, \quad \beta^2 = 1 \quad (\beta = \pm 1, \quad x(t_f) = \beta\nu)
\end{aligned}
\tag{2.4}
$$

The parameter β occurring in the transversality condition has the meaning of a reduced Lagrange multiplier and takes discrete values. Eliminating $x(t_f)$ and σ, we obtain a closed system of equations in η, ν and t_f, μ; the parameter β is also to be determined. Computing the roots and investigating them for arbitrary values of the vectors v^0 and x^0 present a major difficulty in solving the initial optimal control problem. To overcome this difficulty, we use the structural properties of the coefficients $V_{\eta,\nu}(t_f)$ and $X_{\eta,\nu}(t_f)$ as functions of the parameters t_f, μ and σ and reduce system (2.4) to the form

$$
\begin{aligned}
&x^0 = \beta\nu + t_f^2(a_\eta\eta + \mu a_\nu\nu), \quad v^0 = t_f(b_\eta\eta + \mu b_\nu\nu) \\
&a_\eta = b_\eta = \mu^{-2}(a + \sigma b), \quad b_\eta = -b/\mu \\
&a_\eta = -\tfrac{1}{2}\mu^{-3}[(\mu + 3\sigma)a + \mu + (3\sigma^2 - 1)b] \\
&a = q(t)\Big|_{t=0}^{t=t_f} = (1 - 2\mu\sigma + \mu^2)^{1/2} - 1, \quad \mu \geqslant 0, \quad |\sigma| \leqslant 1 \\
&b = V(t)\Big|_{t=0}^{t=t_f} = \ln((\mu - \sigma + (1 - 2\mu\sigma + \mu^2)^{1/2})/(1 - \sigma))
\end{aligned}
\tag{2.5}
$$

The rest of the solution of system (2.5) is analogous to the procedure used in the steering of a point mass to the origin $x(t_f) = 0$ at zero velocity $v(t_f) = 0$ [3]. The essential difference between this case and that considered is the presence of the term $\beta\nu$ in the formula for x in (2.5). One should also note one property of importance for further analysis – the fact that the coefficients $a_{\nu,\eta}$ and $b_{\nu,\eta}$ are independent of the unknown t_f. In addition, in the general case one has strict inequalities $\mu > 0$, $|\sigma| < 1$; the cases in which these are equalities are critical, requiring special investigation and an accurate limiting approach (see below).

3. A NUMERICAL-ANALYTICAL SOLUTION OF THE BOUNDARY-VALUE PROBLEM

The system of transcendental equations (2.5) will be solved by applying elementary algebraic transformations and reducing the system to three equations for t_f, μ and σ. The reduction may be achieved in different ways [3]. One of them is to solve a linear system with block-diagonal matrix for η and ν. This operation is fairly simple, though rather laborious. Thus, suppose the vectors η^* and ν^* are defined as linear functions of the vectors x^0 and v^0 by a matrix with block-diagonal structure. One can then derive three relations for the unknowns t_f, μ, σ: $\eta^{*2} = 1$, $\nu^{*2} = 1$, $(\eta^*, \nu^*) = \sigma$. After determining the roots of this system (taking into account the two possibilities $\beta = \pm 1$), t_{fi}, μ_i and σ_i, one chooses the optimum root

$$t_{fi} \to \min_i, \quad t_{fi} > 0, \quad \mu_i > 0, \quad |\sigma_i| < 1$$

and substitutes it into the expressions obtained for η^*, ν^*, which are used to construct the control and trajectories, see (2.1) and (2.2).

However, this approach is tedious. Therefore, at the first step of the solution of system (2.5) using elementary transformations, we also write down the three equations for t_f, μ and σ in another form

$$\begin{aligned} &l^2 - 1 = 2t_f^2\beta(\sigma a_\eta + \mu a_\nu) + t_f^4(a_\eta^2 + 2\mu\sigma a_\eta a_\nu + \mu^2 a_\nu^2) \\ &h^2 = t_f^2(b_\eta^2 + 2\mu\sigma b_\eta b_\nu + \mu^2 b_\nu^2) \equiv t_f^2 H^2(\mu, \sigma) \\ &c = \beta t_f(\sigma b_\eta + \mu b_\nu) + t_f^3(a_\eta b_\eta + \mu\sigma a_\eta b_\nu + \mu\sigma a_\nu b_\eta + \mu^2 a_\nu b_\nu) \\ &l = |x^0|, \quad h = |v^0|, \quad c = (x^0, v^0) \end{aligned} \tag{3.1}$$

For convenience, by analogy with (1.4), we have introduced in (3.1) scalar parameters l, h and c with an intuitive mechanical meaning. Recall that the n-dimensional system ($n \geqslant 2$) was again equivalent to a two-dimensional system ($n = 2$), which is conveniently represented in the plane. To avoid misunderstandings, we point out that when $n = 2$ the phase space x, v is four-dimensional. We may assume without loss of generality that one of the quantities x_1^0 and x_2^0 is zero.

Our main attention will now be given to finding the roots of system (3.1) of transcendental equations in t_f, μ and σ and analysing them as functions of the known parameters l, h and c, which vary in infinite intervals $l, h \geqslant 0$, $|c| < \infty$. The equations are transcendental because of the presence of power and logarithmic functions, see (2.5). This is what causes the major computational difficulties, since when computations are carried out the required quantities vary in power and logarithmic scales. In addition, the difficulties are aggravated when the initial multidimensional problem is degenerate, that is, when the optimum motion is nearly linear (see below and [3]).

Thus, let us consider the system of equations (3.1) and reduce its order by eliminating the unknown $t_f > 0$. This is conveniently done by using the second relation, which yields a unique expression for t_f in terms of μ and σ

$$t_f = h / H(\mu, \sigma), \quad h, \ H > 0 \tag{3.2}$$

The function H is defined according to (3.1). After substituting (3.2) into the first and third equations of (3.1), we obtain a system of equations in the two unknowns μ and σ.

We introduce a new variable ω by the formula

$$\sigma = \frac{\omega^2 - 2\mu\omega - 1}{(\omega - 1)^2} \tag{3.3}$$

Then the radicand in the definition of a in (2.5) becomes

$$1 - 2\mu\sigma + \mu^2 = \left(\frac{\mu(\omega + 1) + 1 - \omega}{\omega - 1}\right)^2 \tag{3.4}$$

Next, since $\sigma \leqslant 1$, it follows from (3.3) that $\mu\omega \geqslant \omega - 1$. Since $\mu > 0$, it follows that $\mu(\omega + 1) > \omega - 1$. Consequently, the numerator of the fraction in (3.4) takes positive values. On the other hand, $\sigma \geqslant -1$, and by (3.3) we have $\omega(\omega - (\mu + 1)) \geqslant 0$. If $\omega > 0$, we obtain $\omega \geqslant \mu + 1$, which, since $\mu > 0$, implies $\omega > 1$. Thus, if $\omega > 0$, the denominator of the fraction in (3.4) is positive. Then, by (3.4) and the last row in (2.5), we obtain

$$a = \mu(\omega + 1)/(\omega - 1) - 2, \quad b = \ln \omega \tag{3.5}$$

Formula (3.3) enabled us to convert the radical defining the value of a into an expression which is a linear function of μ and a fractional-linear function of ω. At the same time, we have been able to replace the argument of the natural logarithm, which was a function of the unknowns σ and μ, by ω – which is quite unexpected. Unfortunately, attempts to find a simple relation between the behaviour of the parameter ω and the phase variables have proved unsuccessful.

By substituting expression (3.2) and making the change of variable (3.3) we convert the first of equations (3.1) into a fourth-degree equation in the unknown μ. In view of their complexity, the coefficients of this polynomial, which depend only on ω, l, c, h and β, were constructed by computer algebra in the form of program fragments in C++ language. When a numerical simulation was carried out, it transpired that at certain values of these four quantities the roots of the polynomial cannot be determined with satisfactory accuracy, owing to round-off errors. The analysis and transformation of the coefficients were greatly hindered by their cumbersome structure. It turned out, however, that there is a comparatively simple change of variables which not only enables the whole expression to be simplified considerably but also enables one to reduce the influence of computation errors.

Now let us consider negative values of the auxiliary variable ω. Then the denominator of the fraction in (3.4) is negative and the last row in (2.5) gives the following expression for the parameter b instead of (3.5)

$$b = \ln((\mu - \omega + 1)/(\mu\omega - \omega + 1))$$

We introduce a new variable s, expressed in terms of μ and ω as follows:

$$s = \frac{\mu - \omega + 1}{\mu\omega - \omega + 1} \tag{3.6}$$

Introduction of the unknown s enables us to establish an interesting fact. If ω is eliminated by using formulae (3.6) and substituted into the right-hand side of (3.3), the result is

$$\sigma = \frac{s^2 - 2s\mu - 1}{(s - 1)^2}$$

which is identical with (3.3) except that s has replaced ω. Consequently, all the relations obtained above remain valid with s replacing ω. In particular, in this case $b = \ln s$, where $s > 1$. Hence it follows that we need consider only values of $\omega > 1$.

Using formula (3.6) one can express μ in terms of s and ω and henceforth use it regardless of the sign of ω. Indeed

$$\mu = \frac{s\omega + 1 - \omega - s}{s\omega - 1}$$

Substitution of this expression into Eq. (3.1) produces a fourth-degree polynomial in s. Unfortunately, the coefficients are still very cumbersome, but they are nevertheless much more compact than in the case of the polynomial in μ, and, above all, they do not produce the same computational difficulties (except for cases in which $\omega \approx 1$, see below).

The computations were carried out according to the following scheme. As initial data we used the quantities l, c and h. At the first step the third equation of (3.1) was considered as the definition of a certain function $c^*(l, h, \omega, \beta, s(l, h, \omega, \beta))$. Since l and h are given and $\beta = \pm 1$, $c^*(\omega)$ was actually constructed only for $\beta = 1$ and $\beta = -1$. The values of s were computed by Cardano's formulae as the roots of the polynomial resulting from the first equation of (3.1). As a result, up to four branches were obtained for each β on the graph of $c^*(\omega)$. Now, based on the form of the graph, the branches were determined, as were the necessary ranges of values of ω containing the desired roots of the equation $c^*(\omega) = c$. This equation was solved numerically by bisections. If computations with Cardano's formulae are considered as an elementary operation, one in fact has to solve only one transcendental algebraic equation, in one unknown ω. This procedure was carried out for each root of each of the branches, whose existence was established at the previous step. Finally, the solution corresponding to the least duration of motion t_f was chosen.

For all the examples considered, the larger the value of ω, the larger was the duration of the motion, but it is difficult to prove this analytically. In addition, when $\omega \gg 1$, each of the branches is always a monotone function. These facts facilitated the application of the algorithm. On the other hand, when the optimal trajectory is approximately linear, it turns out that the corresponding values of ω are close to unity; the control problem becomes degenerate. The round-off errors arising in that case obstruct normal operation of the program. These difficulties may be overcome in various ways. First, one can apply linear interpolation of the graphs $c^*(\omega)$, since the required value of ω is finite. Second, one can use the exact solution for the case of motion along a straight line, as described in Section 5.

4. DETERMINATION OF THE SOLUTION OF THE OPTIMAL CONTROL PROBLEM

After the optimal value of $\omega = \omega(l, h, c)$ and the corresponding value of $\beta = +1$ ($\beta = \beta(l, h, c)$) had been found for each fixed choice of l, h and c, the following parameters were computed using the explicit analytical formulae

$$t_f = t_f(l,h,c), \quad \mu = \mu(l,h,c), \quad \sigma = \sigma(l,h,c) \tag{4.1}$$

where t_f is the minimum root of system (3.1) (the response time) and μ and σ are the corresponding auxiliary parameters. This also determines the values of the scalar coefficients $a_{\eta,\nu}(l, h, c)$ and $b_{\eta,\nu}(l, h, c)$ in the block-diagonal system of equations (2.5) for the unit vectors η and ν, which is non-degenerate in the general position [3]. Solving this system, we obtain the required block-diagonal expressions for these vectors in terms of the vectors x^0 and v^0

$$\begin{aligned}
&\eta = (-x^0 t_f b_\nu \mu + v^0(t_f^2 a_\nu \mu + \beta))\Delta^{-1} \\
&\nu = (x^0 t_f b_\eta - v^0 t_f^2 a_\eta)\Delta^{-1} \\
&\Delta = t_f b_\eta (t_f^2 a_\nu \mu + \beta) - t_f^3 a_\eta b_\nu \mu \neq 0
\end{aligned} \tag{4.2}$$

The coefficients of the known n-vectors x^0 and v^0, are non-linear functions of these vectors, determined numerically from the values $l = |x^0|$, $h = |v^0|$, $c = (x^0, v^0)$. Substitution of the unit vectors η and ν from (4.2) into the expression (2.1) for u^* yields an n-dimensional time-optimal open-loop control u_p

$$\begin{aligned}
&u_p(t, x^0, v^0) = -k_x(t,l,h,c)x^0 - k_v(t,l,h,c)v^0 \\
&k_x = (|q|\Delta)^{-1}\mu(b_\nu t_f + b_\eta t), \quad |q| \equiv Q(t,l,h,c) \\
&k_v = -(|q|\Delta)^{-1}(t_f(a_\nu t_f + a_\eta t)\mu + \beta), \quad \Delta = \Delta(l,h,c)
\end{aligned} \tag{4.3}$$

The optimal response time t_f and the parameter β are determined by the three quantities l, h and c, using formulae (4.1) and the results of Section 3. Note that the values $\beta = \pm 1$ correspond to the situations in which the "landing" takes place from the inside or the outside, respectively [4]. The corresponding motions may involve no intersections of the sphere, or one or two intersections (see below). The function Q in (4.3) is determined using formula (2.1) and the quantities t_f, μ and σ just found. Substitution of the vector-function u_p from (4.3) into Eqs (2.1) and integration with respect to t yield the trajectory, which may be found analytically in the form of (2.2), (2.3), all the constants t_f, μ, σ and η being defined in terms of the known vectors x^0 and v^0. On the basis of these parameters, various characteristics of the optimal trajectory may be computed; in particular, at $t = t_f$ the angle γ between the tangent to the trajectory and the normal to the surface is determined by the expression

$$\cos\gamma = -(u, x)_{t_f} = -\beta(\sigma - \mu)(1 - 2\sigma\mu + \mu^2)^{-1/2}$$

Thus, we have completely constructed a solution of the problem of the time-optimal open-loop control steering of a point mass from an arbitrary initial state to a sphere in n-dimensional ($n \geqslant 2$) space at zero velocity, by a bounded force. The degenerate case $n = 1$ (or the case of motion along a straight line) will be investigated below, in Section 5.

The above algorithm may be used to construct a feedback time-optimal control $u_s(x, v)$ and the Bellman function $T(x, v)$ (see Section 1). To that end, let us assume that the measurements of phase variables and the computations described in Section 3 may be carried out in practice at each instant of time with sufficient speed, or that it is possible to store and approximate functions of three variables over a sufficiently wide range of measurement. Then a feedback control u_s and function T are defined by

$$\begin{aligned}
&u_s(x, v) = -k_{xs}(l, h, c)x - k_{vs}(l, h, c)v \\
&T(x, v) = t_f(l, h, c), \quad l = |x|, \quad h = |v|, \quad c = (x, v) \\
&k_{xs}(l, h, c) = k_x(0, l, h, c), \quad k_{vs}(l, h, c) = k_v(0, l, h, c)
\end{aligned} \tag{4.4}$$

The scalar factors k_{xs} and k_{vs} in (4.4) have the meaning of feedback coefficients with respect to the position x and velocity v, respectively. Interestingly, the feedback coefficient matrices are diagonal (proportional to identity matrices). Unlike the case of steering a point to the origin [3], these coefficients, as well as the feedback coefficients with respect to the unit vectors x/l and v/l, depend on the three variables l, h and c. This makes their numerical-graphical representation, and the construction of a complete picture of the synthesis, difficult. The feedback control and the corresponding trajectories may be represented as an effective computational procedure (see below, Section 6, for computation results and comments). We also note that the Bellman function $T(x, v)$, defined as a solution of the Cauchy problem (1.3) or (1.4), and the corresponding optimal feedback control $u_s(x, v)$, are computed after finding the optimal root of a system of transcendental algebraic equations and unit vectors (4.2).

5. TIME-OPTIMAL CONTROL IN MOTION ALONG A STRAIGHT LINE

In the critical case, when the optimal control and all the basic relations are degenerate, separate consideration is necessary. As observed, when $n \geq 2$ one obtains one-dimensional motion if, at some time t_0, say $t_0 = 0$, at least one of the following equalities holds

$$x^0 = 0, \quad v^0 = 0, \quad (x^0, v^0) = \pm l^0 h^0 \tag{5.1}$$

Obviously, if $n = 1$ the last equality of (5.1) is true, namely, $(x^0, v^0) = \pm l^0 h^0$. If $|\sigma| = 1$, the vectors η and ν are collinear, and consequently the control vector is collinear with the same straight line at all times. It follows from the transversality condition that this straight line passes through the origin. If the initial velocity vector is not collinear with that line, it can never be made to vanish. Thus, the condition $\sigma = 1$ implies one-dimensional motion. The case $\mu = 0$ has exactly the same implication.

Thus, let us consider the corresponding time-optimal problem for the one-dimensional system

$$\begin{aligned}
&\dot{x} = v, \quad \dot{v} = u, \quad |u| \leq 1, \quad x(0) = x^0, \quad v(0) = v^0 \\
&|x(t_f)| = 1, \quad v(t_f) = 0, \quad t_f \to \min_u
\end{aligned} \tag{5.2}$$

where x, v and μ are scalar variables. According to (5.2), it is required to steer the system from an arbitrary phase point (x^0, v^0) to the point $(+1, 0)$ or $(-1, 0)$ in a minimum time t_f. It is fairly easy to solve this problem using the maximum principle.

The phase plane (x, v) is divided by an antisymmetric curve (separatrix) into two parts $P_{\pm 1}$ (Fig. 1)

$$x = -v(1 + (v/2)^2)^{1/2} \equiv d(v), \quad P_{\pm 1} = \{x, v: \ x \gtrless d(v)\} \tag{5.3}$$

In the unbounded domains $P_{\pm 1}$ the optimal control $u_{\pm 1}$ steers the system from a phase point (x, v) to the state $(\pm 1, 0)$, respectively. The control switching curves have the standard form (Fig. 1)

$$L_{\pm 1} = \{x, v: \ x \pm 1 = \tfrac{1}{2}|v|v, (x, v) \in P_{\pm 1}\} \tag{5.4}$$

It may be shown by simple estimates that the separatrix $x = d(v)$ in (5.3) lies wholly in the region between the switching curves $L_{\pm 1}$ of (5.4) and approaches $L_{\pm 1}$ as symptotically as $v \to -\infty$ and L_{-1} as $v \to \infty$. Each curve L_{+1} consists of two half-branches $L_{+1}^{\pm}$, $L_{-1}^{\pm}$, which reach the points $(\pm 1, 0)$ when

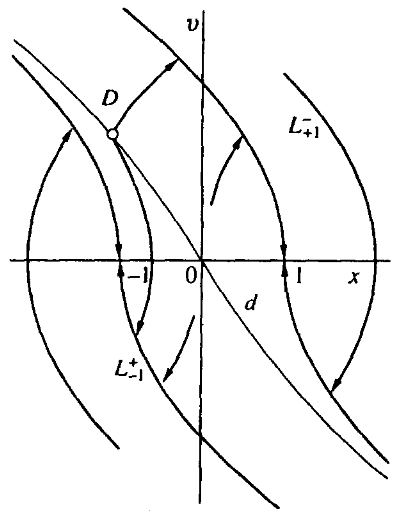

Fig. 1

$u_{+1} = \pm 1$, $u_{-1} = \pm 1$, respectively. The optimal control in each domain $P_{\pm 1}$ changes sign once when the system reaches the phase curve $L_{\pm 1}$. The optimal phase trajectory remains in the domain $P_{\pm 1}$, that is, it never intersects the curve $x = d(v)$. The optimal time $t_f = T_{\pm 1}$ is determined by the formulae

$$T_{+1}(x,v) = \pm v + 2\left(\pm(x-1) + \frac{1}{2}v^2\right)^{1/2}, \quad (x,v) \in P_{+1}, x - 1 \gtrless -\frac{1}{2}|v|v \tag{5.5}$$

$$T_{-1}(x,v) = \pm v + 2\left(\pm(x+1) + \frac{1}{2}v^2\right)^{1/2}, \quad (x,v) \in P_{-1}, x - 1 \gtrless -\frac{1}{2}|v|v$$

On the separatrix $x = d(v)$ the functions T_{+1} and T_{-1} of (5.5) have the same values

$$T_{\pm 1}(d(v),v) = T_d(v) = -v + 2\left(-d(v) + 1 + \frac{1}{2}v^2\right)^{1/2} =$$

$$= v + 2\left(d(v) + 1 + \frac{1}{2}v^2\right)^{1/2}, \quad x = d(v) = -v\left(1 + \left(\frac{v}{2}\right)^2\right)^{1/2} \tag{5.6}$$

Thus, if the phase point D is on the separatrix, the time-optimal problem has not one but two solutions (see Fig. 1). The point will move either to $(+1, 0)$ or to $(-1, 0)$; the time will be determined by (5.6).

Thus, optimal behaviour of the system, optimal control and optimal time for the one-dimensional problem are completely determined (see Fig. 1). By analogy with the standard approach, one can construct an open-loop control and trajectory of the optimal motion. The set of points of singular control has been described (the solution of the problem is not unique). Construction of the singular set in the general case of a two-dimensional system ($n = 2$) would be interesting.

6. RESULTS OF A MATHEMATICAL SIMULATION AND CONCLUSIONS

Whether computations of optimal control and motion utilizing the methods of Section 4 are efficient depends on the solution of various problems for a wide range of initial values of the two-dimensional vectors x^0 and v^0. Theoretically speaking, one is naturally interested in situations in which the initial point $x^0 = (x_1^0, x_2^0)$ is near the circumference ($l \approx 1$) or on it ($l = 1$). Important cases from the applied standpoint are those in which the initial point is far from the circle ($l \gg 1$) or near the origin ($0 < l \ll 1$). Of course, the shape of the optimal trajectories and the control law will depend essentially on the parameters $h = |v^0|$ and $c = (x^0, v^0) = lh \cos\alpha$. As already noted, we may assume without loss of generality that $x_2^0 = 0$, that is, $l = |x_1^0|$.

It would be difficult to describe the set of control modes with exhaustive completeness. We will therefore discuss in detail some typical situations, as illustrated in Figs 2–6 (trajectories) and 7–10 (controls and velocities).

In Fig. 2 we show two families of trajectories, 1–3 and 4–6, the main difference between which is that the ray of the initial velocity vector intersects the disk for the first family ($\alpha = 160°$) but not for the second ($\alpha = -150°$). The starting points are also spaced apart, $x_1^0 = 2$ and $x_1^0 = 3$, though with the same initial coordinate $x_2^0 = 0$; the velocities illustrated are $h = 2.5, 1.1, 1.5, 1, 2, 3$. Depending on the value of h, the circumference may be intersected either once (curve 2), twice (curve 1), or not at all (all the other curves). In addition, the trajectories described are either relatively simple ones "in decelerating mode" (curves 2–5 in Fig. 2; see also Fig. 7, in which the controls $u_{1,2}$ and velocities $v_{1,2}$

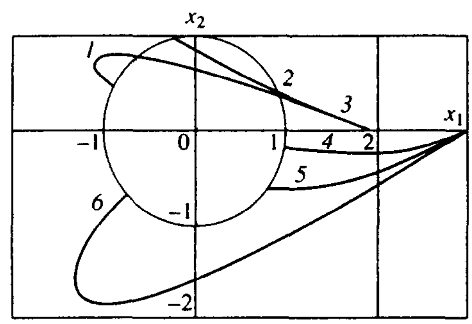

Fig. 2

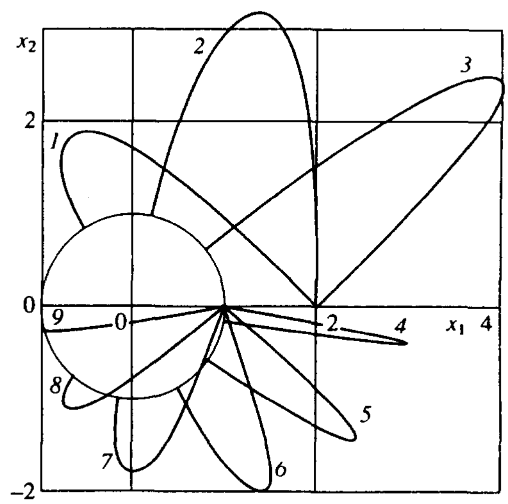

Fig. 3

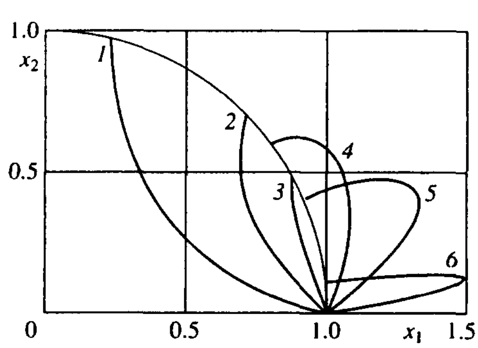

Fig. 4

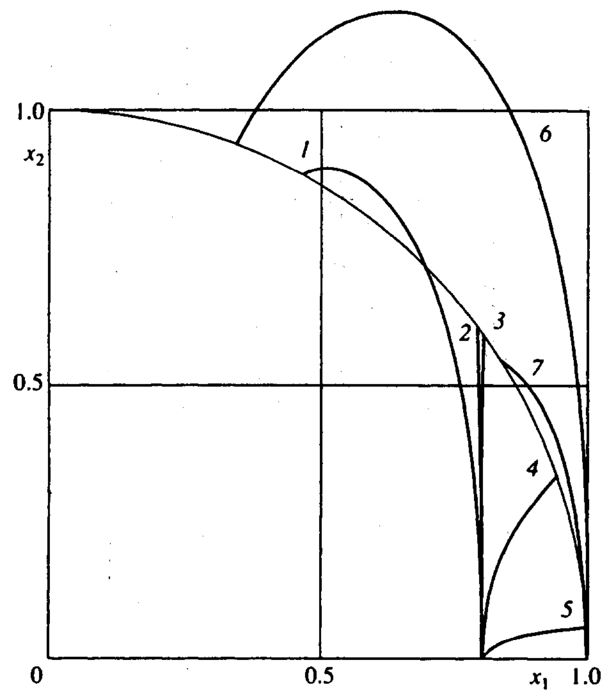

Fig. 5

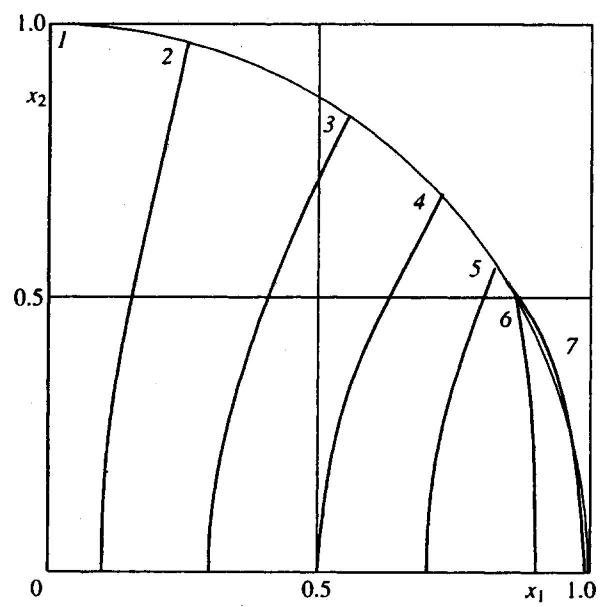

Fig. 6

are shown), or "with re-entry" (curves 1, 6). In Fig. 8 we show the controls and velocities corresponding to curve 6 in Fig. 2, for which $x_1^0 = 3$, $h = 3$, $\alpha = -150°$. "Re-entry" trajectories are accompanied by pronounced intermediate intervals of motion at low velocity ("reversal" in curve 6 and the associated low-velocity region in Fig. 8), and by "control switching" (Fig. 8). The optimal times t_f for trajectories 1–6 are as follows:

$$t_f \approx 3.58, 2.21, 1.50, 2.25, 2.20, 5.06$$

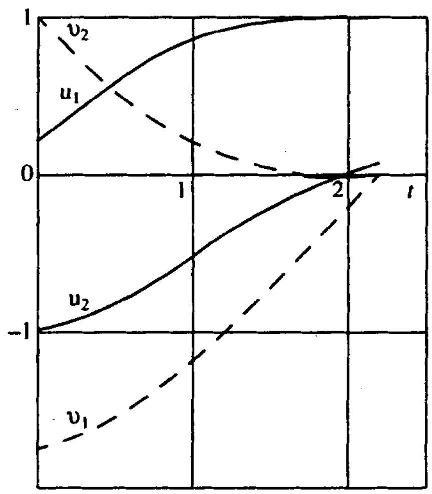

Fig. 7

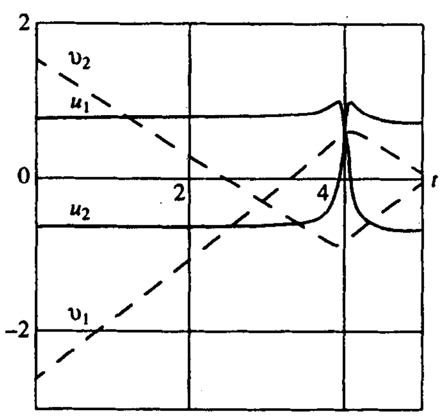

Fig. 8

For trajectories with re-entry, t_f varies more rapidly as h increases than in the case of motion "in decelerating mode".

Similar comments relate to the trajectories and control modes shown in Fig. 3, which again contains two families of curves, 1–3 and 4–9. Here all trajectories involve "re-entry." Curves 1–3 correspond to $x_1^0 = 2$, $h = 2.5$, $\alpha = 135°, 90°, 45°$. Trajectories 4–9 begin on the boundary of the disk at a velocity $h = 2.5$ and different angles

$$\alpha = -10°, -40°, -70°, -110°, -140°, -170°$$

Clearly, trajectories 1–6 do not intersect the disk, while 7–9 do so; these properties are due to the direction and magnitude of the velocity. All landings on the circumference take place from the outside. The times t_f for curves 1–9 are as follows:

$$t_f = 4.36, 5.56, 6.31, 4.82, 4.68, 4.38, 3.73, 3.07, 2.29$$

Examples of trajectories of the most pronounced "re-entry" type are curves 3–5. Besides the "limiting" control modes illustrated in Figs 7 and 8, "intermediate" modes may also appear (Fig. 9).

Figure 4 illustrates a family of trajectories 1–6 for velocity $h = 1$, less than that for curves 4–9 in Fig. 3. Trajectories 4–6, which do not intersect the circumference, are qualitatively similar to curves 1–9 of the "re-entry" type in Fig. 3. Trajectories 1–3, which "land" inside the disk, may be associated with motions "in decelerating mode," see curves 2–5 in Fig. 2 (cf. Also trajectories 3–5 in Fig. 5 and the curves in Fig. 6). The optimal times t_f for trajectories 1–6 are

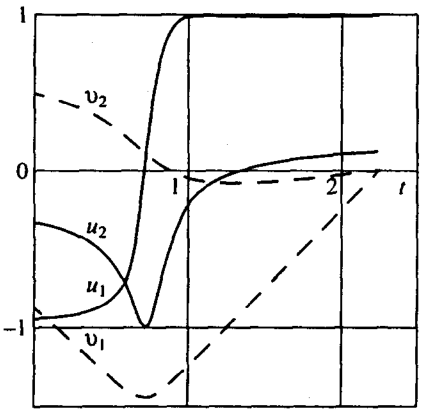

Fig. 9

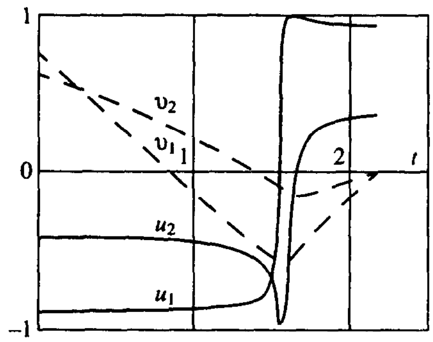

Fig. 10

$$t_f \approx 1.91, 1.37, 1.01, 1.67, 2.19, 2.40$$

It is noteworthy that the optimal time is not a monotone function of the initial angle of inclination of the velocity vector.

It has also been observed that the non-degeneracy property, valid at $t = 0$, remains valid for $0 < t < t_f$.

In Fig. 5 we show two families of curves for which the velocity vector at the starting time $t = 0$ is orthogonal to the x_1 axis, that is, $\alpha = 90°$. For the first family (curves 1–5) $x_1^0 = 0.8$; for the second (curves 6 and 7) $x_1^0 = 1$. In both cases the parameter of the family is the value of the velocity h. For the first family it takes the values $h = 1.3, 1.101, 1.0910, 0.7, 0.1$, and for the second, $h = 1.5, 1$. It is interesting to note that, for the first family, a critical value $h^* \approx 1.1$ exists at which the decelerating mode (see curve 3) becomes a re-entry mode (see curve 2) and the "landing" takes place from the outside. All the curves remain inside the disk at $0 < h < h^*$, intersecting the boundary at $h > h^*$. The optimal response times for curves 1–7 are, respectively

$$t_f = 1.67, 1.10, 1.09, 0.91, 0.89, 2.60, 1.13$$

It should also be noted that the optimal time for curves of the first family is not a monotone function of the velocity. It first somewhat decreases as h increases; for example, at $h = 0.1$ the time is $t_f \approx 0.89$, while at $h = 0.4$ it is $t_f \approx 0.88$. For $h \geqslant 0.5$ the optimal time begins to increase, e.g., $t_f \approx 0.91$ at $h = 0.7$ and $t_f \approx 1.09$ at $h = 1.09$.

In Fig. 6 we show a family of trajectories (curves 1–7) for which $h = 1$, $\alpha = 90°$, $v_1^0 = 0$, $v_2^0 = 1$, while the value of x_1^0 varies in the range $0 < x_1^0 < 1$. Curve 1 corresponds to $x_1^0 = 0.001$ and curve 7 to $x_1^0 = 0.99$; trajectories 2–6 begin at $x_1^0 = 0.1, 0.3, 0.5, 0.7, 0.9$. All the control modes in Fig. 6 correspond to "deceleration". In a strict sense, motion with deceleration takes place at $x_1^0 = 0$. However, already at $x_1^0 = 0.001$ there is a short initial interval ($\Delta t \approx 0.25$) of acceleration ($u_2 \approx 1$, $u_1 \approx 0$) along the x_2 axis, followed by a decelerating section ($u_2 \approx -1, u_1 \approx 0$). At $x_1^0 = 0.1$ (curve 2) there is a short initial interval $\Delta t \approx 0.25$ in which the control u_2 varies almost as in the case of $x_1^0 = 0.001$ (u_2 is slightly less in absolute value); the control u_1 is non-zero and produces a marked shift of the endpoint of the trajectory to the right. It is interesting to observe the behaviour of curve 7 near the boundary of the disk: the optimal trajectory intersects the circumference and landing take place from the outside. Curves 1–7 correspond to the following response times

$$t_f \approx 1.45, 1.44, 1.37, 1.24, 1.09, 1.01; 1.11$$

The following interesting fact should be noted: as the value of x_1^0 approaches the right endpoint of the interval [0, 1], the optimal time decreases relatively quickly (which seems natural) but then increases, which is not at all obvious. This property of the dependence on x_1^0 is analogous to the property considered above, of the dependence on h for trajectories 1–5 in Fig. 5. The fairly common failure of the optimal time to be a monotone function of the parameters is, of course, aggravated by the fact that the terminal set is not a point but a circle.

Thus, the computational results presented above indicate the high efficacy of the algorithm proposed for constructing time-optimal modes of motion in solving a model problem concerning the most rapid soft landing of an object, driven by a bounded force, on a sphere. Investigation of the optimal trajectories yields several interesting qualitative properties of the controlled motions, as observed above.

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